The PAPM with Heterogeneous Preferences and Expectations

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Abstract

The Popularity Asset Pricing Model (PAPM) has similar assumptions to the Capital Asset Pricing Model (CAPM), but different conclusions. In the CAPM, the expected excess return of each security is proportional to a single market systematic risk (beta), while in the PAPM, the expected excess return of each security is linearly related to multiple risk and non-risk characteristics (exposures). Relaxing the assumption of homogeneous forecasts and allowing for heterogeneous forecasts changes most of the CAPM's conclusions but none of the PAPM's conclusions other than introducing another reason that investors hold unique portfolios. With heterogeneous expectations, many investors believe they have skill, but some have more skill than others. These differences in forecasting skill, together with differences in risk and non-risk preferences lead to a variety of different types of market participants. We illustrate the impact of both heterogeneous expectations and popularity preferences on expected returns and portfolios through a series of highly stylized numerical examples. The PAPM with heterogeneous expectations can be viewed as an extension of the CAPM, but it may be more useful to view the CAPM as special case of the PAPM. By incorporating investor preferences and allowing for heterogeneous forecasts, the PAPM takes two major steps towards asset pricing in the real world.

Introduction

In *Popularity: A Bridge Between Classical and Behavioral Finance*, by Ibbotson, Idzorek, Kaplan, and Xiong (2018), henceforth IIKX, Chapter 5 presents a formal version of the Popularity Asset Pricing Model (PAPM) as a direct extension of the Capital Asset Pricing Model (CAPM), but introducing popularity preferences. The chapter starts by presenting the assumptions and conclusions of the CAPM, and then follows the same formal process in the development of the PAPM. An explicit desire was to start with the CAPM, and then to extend the CAPM to the PAPM to account for popularity preferences. As such, the PAPM of IIKX inherited most of the assumptions of the CAPM. One of those assumptions is that market participants share the same (homogeneous) forecasts or capital market assumptions. In this article we examine the implications of relaxing the assumption of homogeneous forecasts, arriving at a more generalized and realistic version of the PAPM in which rather than viewing the PAPM as an extension of the CAPM, the CAPM should be viewed as an elegant special case of the PAPM.

Under the PAPM, some investors may knowingly and willingly give up some return in order to hold portfolios that reflect their unique preferences for certain characteristics, such as more liquidity, a preference for dividends, or a low carbon footprint. IIKX state, "[t]he investors with strong preferences might be called 'the willing losers'…"[1] Aggregate investor preferences drive demand and thus asset prices and result in preference driven popularity premiums. The assumption of homogeneous forecasts allows for those without strong preferences to knowingly profit from those with strong preferences who knowingly sacrifice return for their preferences, but it does not allow for more skilled investors to profit from less skilled investors (beyond pure preferences). Heterogeneous forecasts give rise to additional types of market participants, beyond the knowing winner/losers caused by differing preferences. Market participants with more relative skills (with similar preferences) are expected to outperform those with less relative skill. Thus, heterogeneous expectations give us an additional reason to have winners/losers in the PAPM.

We begin with a review of the assumptions and conclusions of the CAPM and PAPM with homogeneous expectations, and then examine the implications of heterogeneous forecasts. Next we provide a select literature review and contrast the PAPM with several important areas of asset pricing research prior to proceeding with a formal presentation of the CAPM and PAPM with heterogeneous expectations. Finally, we present an illustrative example in which we compare/contrast four models: the CAPM with homogeneous expectations, the CAPM with heterogeneous expectations, PAPM with homogeneous expectations, and the PAPM with heterogeneous expectations.

[1] The phrase "willing loser" comes from a quote of Robert Arnott in Rostad (2013).

CAPM and PAPM Assumptions and Conclusions with Homogeneous Forecasts

With some rearranging relative to IIKX for the purposes of comparing assumptions, Exhibit 1 presents the assumptions of the CAPM and PAPM side-by-side. Sequentially, the first three assumptions are the same or nearly the same across the two models. The first assumption in Exhibit 1 is the key assumption that we will be examining: homogeneous forecasts. The PAPM is less restrictive and differs from the CAPM by assuming that security characteristics unrelated to risk matter and are important to investors with heterogeneous *preferences* that go beyond risk.

Exhibit 1: Model Assumptions

CAPM – Homogeneous		**PAPM – Homogeneous**
All investors have the same forecasts; that is, the same capital market assumptions (expected returns, standard deviations, and correlations).	=	All investors have the same forecasts; that is, the same capital market assumptions (expected returns, standard deviations, and correlations).
All investors can borrow and lend at the same risk-free rate without limit.	=	All investors can borrow and lend at the same risk-free rate without limit.
All investors use Mean-Variance Optimization (MVO) as described by Markowitz (1952, 1959, 1987) to select their portfolios.	≈	All investors use a generalized form of Mean-Variance Optimization (MVO) that linearly incorporates their preferences regarding security characteristics.
Taxes, transaction costs, and other real-world considerations can be ignored.	≠	Investors have preferences regarding these characteristics in addition to their preferences regarding risk and expected return.
N/A	≠	All investors agree on what the characteristics of the securities are.

Moving to the conclusions, in Exhibit 2 we once again attempt to arrange what one might think of as corresponding model conclusions (even though the conclusions themselves are very different between the two models).

Exhibit 2: Model Conclusions

CAPM – Homogeneous		PAPM – Homogeneous
From among all possible portfolios of risky assets, the market portfolio (that is, the cap-weighted combination of all risky assets in the market) maximizes the Sharpe ratio; hence, it is on the efficient frontier.	≠	The market portfolio does not maximize the Sharpe ratio among all portfolios of risky assets.
Each investor combines the market portfolio with long or short positions in the risk-free asset (cash). Hence, investors do not actually need to perform MVO to construct optimal portfolios.	≠	Each investor forms a customized portfolio of the risky assets that reflects his or her attitudes toward each security characteristic. This portfolio is combined with long or short positions in the risk-free asset. MVO is required to construct the overall investor-specific portfolio.
The expected excess return of each security is proportional to its systematic risk with respect to the market portfolio (beta).	≠	The expected excess return of each security is a linear function of its beta and its popularity loadings, which measure the popularity of the security based on its characteristics relative to those of the beta-adjusted market portfolio.

Despite starting with very similar assumptions in Exhibit 1, we see in Exhibit 2 that the conclusions of the CAPM and PAPM are quite different from one another. The market portfolio **is / is not** the Sharpe maximizing portfolio. Mean-variance optimization **is not / is** necessary to arrive at an investor's ideal portfolio. The expected excess return of each security **is** proportional to its systematic risk relative to a single market with respect to the market portfolio (beta), while in the PAPM, excess return **is not** limited to systematic risk, rather the expected excess return of each security is linearly related to multiple risk and non-risk characteristics (exposures). Hence, in the equation for the expected return of a given security, in addition to the beta term, there are terms consisting of the security's popularity loadings multiplied by popularity premiums. The popularity premiums are aggregations of the preferences of the investors regarding the characteristics that investors care about besides risk. In this way, the market aggregates investor preferences in determining the influence of security characteristics on the expected returns and prices of the securities. The equation for expected return in the PAPM is the foundation for our primary question and model extension/generalization in this paper.

What happens to the conclusions of the CAPM and PAPM if we relax the assumption that all investors share the same forecasts (while maintaining the other assumptions)?

The quick answer is that the CAPM's conclusions regarding individual investor portfolios do not hold while conclusions regarding security expected returns do on average.

For the PAPM, the conclusions largely continue to hold, although investor portfolios differ from one another for an additional reason – heterogenous forecasts.

As we shall see, it also becomes evident, that rather than viewing the PAPM as an extension of the CAPM as framed in IIKX, the PAPM with heterogeneous expectations is a more generalized, realistic asset pricing model that allows for unique characteristic preferences and that the CAPM with or without heterogeneous expectations represents more restrictive special cases of the more encompassing PAPM.

The Implication of Heterogeneous Forecasts

In a CAPM-world, if market participants all have different forecasts or opinions on what capital market assumptions to make, after applying mean-variance optimization, all investors will arrive at different efficient frontiers and ultimately hold unique portfolios, rather than the market portfolio. Each investor's different frontier will lead to a different Sharpe ratio maximizing tangency portfolio that they can lever / lend to meet their particular risk appetite. For each investor, the expected excess return of each security is proportional to its systematic risk with respect to the *tangency* portfolio unique to the investor, rather than the *market* portfolio; thus, no investors will hold the market portfolio and all investors will hold a custom portfolio, which can include long / short security positions. Each investor performs mean-variance optimization to arrive at the ideal custom portfolio.

With investors holding a variety of different portfolios, there will be winners and losers due to the relative superiority of forecasts, whether relative expected performance is due to skill or preferences. In aggregate, the asset-weighted forecasts and the portfolio weights lead back to the average market expected return and the market portfolio.

In a PAPM-world, if market participants all have different forecasts or opinions on capital market assumptions and *other preferences*, after applying a generalized form of mean-variance optimization, all investor will arrive at different generalized efficient frontiers and ultimately hold unique portfolios rather than the market portfolio. Each investor's different generalized frontier will lead to a different utility maximizing portfolio that incorporates the investor's preferences for the various security characteristics and their unique forecasts. No investors will hold the market portfolio; rather, all investors will hold a custom portfolio.[2] Of course, the market portfolio exists because all assets are held.

[2] In principle, an investor with diverse expectations and typical preferences might hold the market portfolio since this investor does not have expectational opinions and does not participate in the pricing of assets.

In a PAPM-world, popularity premiums reflect the aggregation of the preferences of the investors regarding the characteristics. With investors holding a variety of different portfolios, there will be winners and losers, which is due to some combination of differences in forecasts and differences in preferences. As in the original PAPM, the market aggregates investor preferences in determining the influence of security characteristics on expected returns. Investors without popularity preferences are more likely to be winners.

Throughout IIKX, when discussing part of the rationale for popularity premiums, it discusses the notion of "willing losers" or, and in many cases, "unknowing losers." Willing losers are those with strong preferences and as a result of those preferences are willing to give up some level of expected return in order to hold a portfolio that reflects their preferences. Willing losers are consistent with the PAPM presented in Chapter 5 of IIKX, specifically the assumption that all market participants share the same forecasts or capital market assumptions. With heterogeneous forecasts we can have a variety of additional winners and losers, due to positive / negative relative skill, which are unrelated to risk and non-risk preferences.[3]

In the real world, mispricing is sometimes observed, but not relative to an unknown true price. For example, when one security is a derivative of another, the law of one price can be violated, yet we still do not know the true price.[4] In the PAPM, investors are not constrained to be rational. Prices are determined by aggregating across investors' heterogeneous expectations however they are formed. Thus, the PAPM does not require a true price. Nevertheless, the more skilled investors are expected to outperform the less skilled investors, since the less skilled investors may not possess, or may not be using information appropriately. Heterogeneous forecasts likely lead to inefficient markets, since the more skilled investors might not be able to offset the price impact of the less skilled investors. While an all-knowing Economist is not part of the PAPM, by including one in our later illustrations, we can insert ending expected values into our analysis, and more easily demonstrate how portfolios and prices are formed as well as examine the degree of inefficiency in the market under various scenarios. We define mispricing as the difference between the price and the unknown true price.

Contrasting the PAPM with other Asset Pricing Models

The PAPM expands and generalizes the CAPM of Sharpe (1964), Lintner (1965), Treynor (1962), and Mossin (1966). While unique, the models presented in this paper – the CAPM with

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[3] Expected winners and losers are due to differences in forecasting skill. Ex ante, those with greater skill and similar preferences are expected to outperform those with less skill. Ex post, winners and losers are the result preferences, relative skill, and random luck.

[4] A well-known example of this occurred in 2000, when 3Com sold off 5% of subsidiary Palm in an initial public offering, while retaining ownership of 95% of Palm; yet, the stock market value of 3Com was significantly lower than 3Com's holdings of Palm.

heterogeneous expectations and the PAPM with heterogeneous expectations – are most closely linked to those of Lintner (1969), in that the heterogeneous expectations are exogenous. The PAPM presented here has both heterogenous popularity preferences and expectations, which leads to equilibrium prices that are functions of both the weighted averages of expectations and the weighted averages of preferences.[5] The "popularity framework" in which investor preferences drive demand and ultimately asset prices was developed in Ibbotson and Idzorek (2014) and Idzorek and Ibbotson (2017), building upon and providing additional rationale for the New Equilibrium Theory (NET) of Ibbotson, Diermeier, and Siegel (1984). IIKX continues to develop the framework, offers a popularity-based explanation for many well-known premiums/anomalies, presents new empirical evidence, and presents of a formal version of the PAPM as a direct extension of the CAPM.[6]

The popularity framework and the PAPM is similar to many of the ideas, discoveries, and models of behavioral finance. The idea of popularity is most closely linked to affect (see Zajonc (1980)), although affect seems to ignore the rational NET oriented preferences. The PAPM is distinct but similar in spirit to the behavioral asset pricing models of Shefrin and Statman (1994), Shefrin (2008), Statman, Fisher, and Anginer (2008), Statman (2017), and Luo and Subrahmanyam (2019).

While we do not rely on the heterogeneous expectations literature beyond Lintner (1969) for this paper, we summarize parts of it here to contrast the PAPM with other models.[7] Lintner (1969) was the first to extend the CAPM to allow for heterogeneous expectations. Williams (1977) expressed what Lintner did in a continuous time framework.

An important area of research focuses on information aggregation in a rational world in which market prices are endogenous to the models. For example, Grossman and Stiglitz (1980) extend the concept of a noisy rational expectations equilibrium introduced by Lucas (1972) in which prices inform endogenous beliefs. Their model includes both informed and uninformed participants with different expectations concluding that because information comes at a cost, market prices do not reflect all information, because if they did, no one would pay for the information in question –the so-called Grossman-Stiglitz Paradox. As such, an informationally efficient market is impossible, even in a rational world. Similarly, Diamond and Verrecchia (1981) assume that prices are rationally determined from each investor having differing noisy information (leading to different expectations). They derive an equilibrium that is not fully revealing of the information, despite the rationality assumption since their result is dependent upon who knows what.

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[5] This aggregation is consistent with works such as Verrecchia (1980) and Rubinstem (1975), although we stop short of concluding that the market consensus is a statement of market information efficiency.

[6] Paul Kaplan deserves primary credit for Chapter 5 of IIKX.

[7] While we believe it is obvious that investors have heterogeneous forecasts, Shefrin (2008) reviews, and in some cases reaffirms, empirical evidence of heterogeneity from experiments presented in Grether (1980), Kahneman and Tversky (1979), De Bondt (1993), and Welch (2000).

Behavioral finance research continues to focus on heterogeneous expectations and the aggregation of information in the development of endogenous asset pricing models, dropping the assumption of rationality. Examples of this general path of research include Cutler, Poterba, and Summers (1990), DeLong et al. (1990), and Hong and Stein (1999). More recently, Barberis et al. (2013) puts forth an extrapolative capital asset pricing model in which some investors may irrationally extrapolate beliefs about future price changes based on past price changes, while other investors may hold fully rational beliefs.

In the CAPM and PAPM, expected ending values are exogenous. Since we do not specify how investors form their expectations, one could presume that some investor expectations are formed by weighting investor initial conjectures with information from market prices, similar to Diamond and Verrecchia (1981) while perhaps others may form expectations in a manner consistent with the extrapolative capital pricing model of Barberis et al (2013). However, following these approaches would lead to much more complex models. Ultimately, like Lintner (1969) and Levy, Levy, and Benita (2006) we do not assume that investors make use of the information revealed by prices Rather, we assume that the source of heterogeneous beliefs could include different interpretations of the same information and unique information available to different investors.

In general, many of these asset pricing models have attempted to address relatively specific issues resulting in relatively complex models of asset prices. Ultimately, we do not assume that expectations and popularity preferences are necessarily rationally formed based on market prices and other information received by investors. In most cases these more specific models are not in contrast with our very general model. As such our models are simpler, more flexible, require fewer assumptions, and relative to the rational models, more closely aligned with behavioral finance.

Superficially, the PAPM is similar to other multifactor asset pricing models, most notably arbitrage pricing theory (APT) of Ross (1976); thus, it is worth noting some of the critical distinctions. First and foremost, while APT and PAPM may appear to be similar, their underpinnings are very different – APT is an arbitrage model while the PAPM, like the CAPM and the other models mentioned above, is an equilibrium model. The practical distinction between arbitrage and equilibrium is in what the two approaches say about prices. Arbitrage models describe the relationships between security prices in a law of one price world, but do not explain *where* prices ultimately come from. APT says the expected return on a security is a linear function of exposures to a set of risk factors, with risk premiums on the factors as the coefficients. It does not, however, explain where the risk premiums come from. In contrast, equilibrium models, such as the CAPM and PAPM explain / derive security prices in terms of both the characteristics of the securities and the preferences of investors. In the case of the PAPM, the premiums may or may not be related to risk. In the PAPM, aggregate investor preferences create the endogenous popularity premiums, while APT factors are exogenous and APT risk premiums are not related to investor preferences.

Finally, as we shall see in the following sections, the PAPM is unique relative to all of these other models mentioned in that we develop it in the familiar context of the CAPM.

The CAPM and PAPM with Homogeneous and Heterogeneous Expectations

The standard CAPM with homogeneous forecasts leads to very special conclusions: the market portfolio is mean-variant efficient; all investors hold a portion of their wealth in the market portfolio; and mean-variance optimization is unnecessary. The model does not result in losers and winners. The market is informationally efficient and there is no mispricing.

The CAPM with heterogeneous expectations only considers systematic risk; yet, due to heterogeneous forecasts all investors form unique portfolios. There are no losers and winners due to preferences as the only investor preference is risk aversion. The market is less than perfectly efficient and there is mispricing; thus, there are winner and losers due to differences in skill.

The PAPM with homogenous expectations, the CAPM with heterogeneous expectations, and the standard CAPM with homogeneous expectations are all special cases of the PAPM with heterogeneous expectations. These three special cases simply represent more restrictive, less realistic assumptions that give rise to fewer types of expected winners and losers.

As an additional special case it is possible that the net attitude towards characteristics is zero, so that the CAPM equation for expected excess returns still prevails and the market portfolio is mean-variance efficient with market average expected returns. But even in that case, investors still tilt their portfolios towards the characteristics that they like and away from the ones that they dislike.

The PAPM with heterogeneous forecasts, or expectations, is the most general and least restrictive (most realistic) of the asset pricing models we have discussed. In a PAPM world with heterogeneous expectations, markets are less than perfectly efficient and there is mispricing.

Prior to a numerical example, in the next three sections we formally present the CAPM and the PAPM with heterogeneous expectations, and then recast these two models in terms of market values and the real economy. In these models, different investors can have different levels of risk aversion, forecasting skill, and non-risk preferences. As for forecasting skills, most investors of all skill levels believe themselves to be skilled, but our assumption is that some investors are more skilled than others. Following the flow of Chapter 5 of IIKX, we start with the CAPM and then move to the PAPM.

Formal Presentation of the CAPM with Heterogeneous Expectations

In our models, we aggregate the demand from the heterogeneous expectations of investors.

While the details of our models differ from those of Lintner (1969), the idea is fundamentally the same; namely, we allow investors to hold different opinions on expected values of key random variables. Hence every investor is on the margin; thus, there is no marginal investor. In the CAPM with heterogeneous expectations, the key random variables are expected returns on individual securities. Hence, investor *i*'s problem is:[8]

$$\max_{\vec{\boldsymbol{x}}_i} U_i(\vec{\boldsymbol{x}}_i) = \vec{\boldsymbol{\mu}}_i{}'\vec{\boldsymbol{x}}_i - \frac{\lambda_i}{2}\vec{\boldsymbol{x}}_i{}'\boldsymbol{\Psi}\vec{x}_i \qquad (1)$$

where

n = the number of risky securities in the market
$\vec{\boldsymbol{\mu}}_i$ = the *n*-element vector of expected security excess returns reflecting investor *i*'s views[9]
$\boldsymbol{\Psi}$ = the $n \times n$ variance-covariance matrix of returns on the risky securities
$\vec{\boldsymbol{x}}_i$ = the *n*-element vector of investor *i*'s allocations (portfolio weights) to the risky securities[10]
λ_i = the risk aversion parameter of investor *i*

Based on investor *i*'s forecasts $\vec{\boldsymbol{\mu}}_i$ and risk aversion coefficient λ_i, investor *i* seeks to maximize utility.

From the first-order condition, we have:

$$\vec{\boldsymbol{\mu}}_i = \lambda_i \boldsymbol{\Psi}\vec{\boldsymbol{x}}_i \qquad (2)$$

Solving for $\vec{\boldsymbol{x}}_i$, we have:

$$\vec{\boldsymbol{x}}_i = \frac{1}{\lambda_i} \boldsymbol{\Psi}^{-1}\vec{\boldsymbol{\mu}}_i \qquad (3)$$

In other words, from equations 2 and 3, we can start with either an investor's portfolio holdings or expected excess returns and solve for the other.

Let
m = the number of investors
w_i = the fraction of wealth held by investor *i;* $\sum_{i=1}^{m} w_i = 1$

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[8] Note the utility function is quadratic in portfolio weights, leading to the optimal portfolio weights being linear in expected returns.
[9] By excess returns, we mean in excess of the return on a risk-free security.
[10] There is a risk-free security to which investor *i* allocates $1 - \sum_{j=1}^{n} x_{ij}$.

Aggregating across investors, in equations 4, 5, and 6, we have the market average level of risk aversion (λ_M), market-weighted average of investor expected security excess returns ($\vec{\boldsymbol{\mu}}_M$), and the market portfolio ($\vec{\boldsymbol{x}}_M$) consisting of the weighted aggregation of the security weights of the investors:

$$\lambda_M = \frac{1}{\sum_{i=1}^{m} \frac{w_i}{\lambda_i}} \tag{4}$$

$$\vec{\boldsymbol{\mu}}_M = \lambda_M \sum_{i=1}^{m} \frac{w_i}{\lambda_i} \vec{\boldsymbol{\mu}}_i \tag{5}$$

$$\vec{\boldsymbol{x}}_M = \sum_{i=1}^{m} w_i \vec{\boldsymbol{x}}_i \tag{6}$$

Aggregating equation 3 across investors, in which we solve for each investor's portfolio based on their expectations, we have the asset-weighted average holdings (the market portfolio):

$$\vec{\boldsymbol{x}}_M = \frac{1}{\lambda_M} \boldsymbol{\Psi}^{-1} \vec{\boldsymbol{\mu}}_M \tag{7}$$

So that:

$$\vec{\boldsymbol{\mu}}_M = \lambda_M \boldsymbol{\Psi} \vec{\boldsymbol{x}}_M \tag{8}$$

Equation 9 decomposes the right side of equation 3, to show that each investor's portfolio differs from the market portfolio due to the difference between each investor *i*'s expected security excess returns and the market-weighted *average* security excess returns:

$$\vec{\boldsymbol{x}}_i = \frac{\lambda_M}{\lambda_i} \vec{\boldsymbol{x}}_M + \frac{1}{\lambda_i} \boldsymbol{\Psi}^{-1} (\vec{\boldsymbol{\mu}}_i - \vec{\boldsymbol{\mu}}_M) \tag{9}$$

Note that the first term on the right-hand side is same as in the standard CAPM, the fraction of the market average portfolio owned by investor *i*. The second term on the right-hand side pinpoints that the uniqueness of investor *i*'s portfolio is driven by the difference between the investor's expected excess returns and the market average excess returns. Note the similarity of equation 9 to the Black-Litterman (1992) model in which an investor's views about expected returns on assets are combined with market consensus views to arrive at an investor's expectations and ultimately their specific portfolio.

The expected excess return on the market portfolio remains:

$$\mu_M = \vec{\boldsymbol{x}}_M{}' \vec{\boldsymbol{\mu}}_M \tag{10}$$

Multiplying equation 8 through by $\vec{\boldsymbol{x}}_M'$, yields:

$$\mu_M = \lambda_M \sigma_M^2 \tag{11}$$

where $\sigma_M^2 = \vec{\boldsymbol{x}}'_M \boldsymbol{\Psi} \vec{\boldsymbol{x}}_M$, which is the variance of the market portfolio.

Rearranging equation 11, it follows that:

$$\lambda_M = \frac{\mu_M}{\sigma_M^2} \quad (12)$$

Thus, λ_M, the average level of risk aversion, identifies the units of excess return per unit of market variance. Substituting the right-hand side of equation 12 for λ_M in equation 8, and rearranging terms, yields the familiar CAPM equation for security expected *excess* returns, but at the *aggregate* market level:

$$\vec{\boldsymbol{\mu}}_M = \vec{\boldsymbol{\beta}}_M \mu_M \quad (13)$$

where the vector of betas is the covariance of each security with the market portfolio divided by the market variance

$$\vec{\boldsymbol{\beta}}_M = \frac{\boldsymbol{\Psi} \vec{x}_M}{\sigma_M^2} \quad (14)$$

We subscript the beta vector with *M* to make it clear that these are betas with respect to the market average portfolio and not the betas that we could define with respect to each investor's tangency portfolio.

As we discuss below, the equilibrium market *values* of the securities reflect the aggregation of the expected ending values across all investors. Expected excess returns, the variances and covariances of returns are all inversely proportional to market values. In the CAPM equilibrium, market values are such that equation 13 holds.

Formal Presentation of the PAPM with Heterogeneous Expectations

Moving to the PAPM, let

p = the number of popularity characteristics
$\boldsymbol{C}$ = $n \times p$ matrix of characteristic exposures of the securities
$\vec{\boldsymbol{\phi}}_i$ = p-element vector of investor i's attitudes toward the characteristics
(The elements can be positive, negative, or zero.)[11]

Investor *i*'s problem is:

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[11] If security *j* has a 100% exposure to a characteristic, a specified attitude of 1% would have the same effect as increasing investor *i*'s expected return up by 1 percentage point. In a subsequent paper, "Implementing the PAPM in Practice," we provide further guidance on implementing the PAPM.

$$\max_{\vec{\boldsymbol{x}}_i} U_i(\vec{\boldsymbol{x}}_i) = \vec{\boldsymbol{\mu}}_i{}'\vec{\boldsymbol{x}}_i + \vec{\boldsymbol{\phi}}_i{}'\boldsymbol{C}'\vec{\boldsymbol{x}}_i - \frac{\lambda_i}{2}\vec{\boldsymbol{x}}_i{}'\boldsymbol{\Psi}\vec{\boldsymbol{x}}_i \tag{15}$$

Relative to equation 1, equation 15 contains an additional term (middle term on the right-hand side) that captures popularity characteristics and investor *i*'s additional preferences. Investor *i*'s preferences for different characteristics can be driven by expectations around popularity premiums or non-return related preferences.

From the first-order condition, we have:

$$\vec{\boldsymbol{\mu}}_i = \lambda_i \boldsymbol{\Psi}\vec{\boldsymbol{x}}_i - \boldsymbol{C}\vec{\boldsymbol{\phi}}_i \tag{16}$$

The solution is:

$$\vec{\boldsymbol{x}}_i = \frac{1}{\lambda_i}\boldsymbol{\Psi}^{-1}\left(\vec{\boldsymbol{\mu}}_i + \boldsymbol{C}\vec{\boldsymbol{\phi}}_i\right) \tag{17}$$

Aggregating equation 17 across investors, we have the security weights of the market portfolio:

$$\vec{\boldsymbol{x}}_M = \frac{1}{\lambda_M}\boldsymbol{\Psi}^{-1}(\vec{\boldsymbol{\mu}}_M + \boldsymbol{C}\vec{\boldsymbol{\pi}}) \tag{18}$$

where $\vec{\boldsymbol{\mu}}_M$ is as defined in equation 5 and

$$\vec{\boldsymbol{\pi}} = \lambda_M \sum_{i=1}^{m} \frac{w_i}{\lambda_i}\vec{\boldsymbol{\phi}}_i \tag{19}$$

$\vec{\boldsymbol{\pi}}$ is the *p*-element vector of aggregate, wealth and risk aversion-weighted investor preferences for different characteristics, and for reasons that will become apparent below, we call $\vec{\boldsymbol{\pi}}$ the vector *of popularity premiums*. Combining the matrix of security characteristics with the vector of popularity premiums, $\boldsymbol{C}\vec{\boldsymbol{\pi}}$ leads to a *n*-element vector of popularity-based adjustments that augment the market expected returns and impact the market portfolio.

From equations 17 and 18, we derive an equation for the portfolio decision of each investor relative to the market portfolio:

$$\vec{\boldsymbol{x}}_i = \frac{\lambda_M}{\lambda_i}\vec{\boldsymbol{x}}_M + \frac{1}{\lambda_i}\boldsymbol{\Psi}^{-1}\left[(\vec{\boldsymbol{\mu}}_i - \vec{\boldsymbol{\mu}}_M) + \boldsymbol{C}\left(\vec{\boldsymbol{\phi}}_i - \vec{\boldsymbol{\pi}}\right)\right] \tag{20}$$

Hence, both differences in expected returns and differences in popularity preferences impact individual portfolio construction.

Solving equation 18 for $\vec{\boldsymbol{\mu}}_M$ yields:

$$\vec{\boldsymbol{\mu}}_M = \lambda_M \boldsymbol{\Psi}\vec{\boldsymbol{x}}_M - \boldsymbol{C}\vec{\boldsymbol{\pi}} \tag{21}$$

Multiplying equation 21 through by $\vec{\boldsymbol{x}}'_M$ yields:

$$\mu_M = \lambda_M \sigma_M^2 - \vec{\boldsymbol{c}}_M{}'\vec{\boldsymbol{\pi}} \quad (22)$$

where $\vec{\boldsymbol{c}}_M = \boldsymbol{C}'\vec{\boldsymbol{x}}_M$, which is the vector of security characteristic exposures of the market portfolio.

From equation 22, it follows that:

$$\lambda_M = \frac{\mu_M + \vec{\boldsymbol{c}}_M{}'\vec{\boldsymbol{\pi}}}{\sigma_M^2} \quad (23)$$

Substituting the right-hand side of equation 23 for λ_M in equation 21, and rearranging terms, yields the generalization of the CAPM equation for market average expected excess returns:

$$\vec{\boldsymbol{\mu}}_M = \vec{\boldsymbol{\beta}}\mu_M + \left(\vec{\boldsymbol{\beta}}\vec{\boldsymbol{c}}'_M - \boldsymbol{C}\right)\vec{\boldsymbol{\pi}} \quad (24)$$

In equation 24 the first term on the right-hand is the same as the right-hand side of the CAPM. The second term represents the impact of popularity on security expected returns. This equation looks like a multifactor asset pricing model, but with the popularity premiums rather than risk premiums. For an individual security *j*, let

$$\delta_{jk} = \beta_j c_{Mj} - C_{jk} \quad (25)$$

so we can write:

$$\mu_{Mj} = \beta_j \mu_M + \sum_{k=1}^{p} \delta_{jk} \pi_j \quad (26)$$

We call δ_{jk} security *j*'s *popularity loading* on characteristic *k*. It is positive if security *j*'s exposure to characteristic *k* is less than that of the beta-adjusted market portfolio and negative if the reverse is true. In this way, a popularity loading of a security is positive for a given characteristic if the security is unpopular with respect to the characteristic and negative if it is popular.

As mentioned early, the PAPM can look similar to APT, in which returns are a linear function of factor/characteristic exposures. For the APT, the linear relationship between expected return and premiums follows directly from the assumption that security returns have a linear relationship to the risk factors. In contrast, in the CAPM and PAPM, the linear structure of expected returns originates from the assumption that the utility derived from the portfolio holdings is quadratic. For the PAPM, this is simplifying assumption that could be dropped.

The Real Economy, Market Values, and Equilibrium

Thus far, we have presented the CAPM and the PAPM with heterogeneous expectations in terms of excess returns on securities. In this section, we recast these models in terms of market values and the real economy so that we can describe the conditions for equilibrium. This sets the stage for the numerical example in the next section.

In one-period models, such as the CAPM and PAPM, the excess return on a given security *j* is:

$$\tilde{r}_j = \frac{\tilde{y}_j}{v_j} - 1 - r_f \qquad (27)$$

where

$\tilde{r}_j$ = the random excess return on security *j*
$\tilde{y}_j$ = the random end-of-period value of security *j*
v_j = the beginning-of-periods market value of security *j*
r_f = the risk-free rate

As done in IIKX, we call the distribution of the random vector of end-of-period values for all risky securities, $\tilde{\boldsymbol{y}}$, the *real economy*. The real economy has two sets of parameters:

1) The expected value of $\tilde{\boldsymbol{y}}$, $E[\tilde{\boldsymbol{y}}]$
2) The variance-covariance matrix $\tilde{\boldsymbol{y}}$, $\boldsymbol{\Omega}$.

With heterogeneous expectations, each investor can have a personal version of these parameters. For simplicity, within this article we assume the investors only differ in the expected values. Let $E_i[\tilde{y}_j]$ denote investor *i*'s expected value of security *j*, be $\tilde{y}_j$. Investor *i*'s expected excess return on security *j* is:

$$\mu_{ij} = \frac{E_i[\tilde{y}_j]}{v_j} - 1 - r_f \qquad (28)$$

The element of the variance-covariance matrix of *returns*, Ψ, for securities *j* and *k* is related to the variance-covariance matrix of the *real economy*, $\boldsymbol{\Omega}$, and market values, v_j, v_k, as follows:

$$\Psi_{jk} = \frac{\Omega_{jk}}{v_j v_k} \qquad (29)$$

Taking market values as given, each investor calculates expected excess returns and the variance-covariance matrix of returns using equations 28 and 29. Then each investor finds an optimal portfolio using equation 3 under the CAPM or equation 17 under the PAPM.

Let

$x_{ij}(\vec{\boldsymbol{v}})$ = investor *i*'s portfolio allocation to security *j* as a function of market values, $\vec{\boldsymbol{v}}$
$q_{ij}(\vec{\boldsymbol{v}})$ = number of shares of security *j* demanded by investor *i* as a function of market values, $\vec{\boldsymbol{v}}$

Investor *i*'s demand for shares of security *j* is:

$$q_{ij}(\vec{\boldsymbol{v}}) = w_i \frac{x_{ij}(\vec{v})}{v_j} \sum_{k=1}^{n} v_k \quad (30)$$

The total demand for shares of security *j* is:

$$q_{Mj}(\vec{\boldsymbol{v}}) = \sum_{i=1}^{m} q_{ij}(\vec{\boldsymbol{v}}) \quad (31)$$

As before the subscript *M* indicates the market average values.

Under our assumptions, the supply of shares of each security is 1. Hence, in equilibrium, market values are such that for each security *j*,

$$q_{Mj}(\vec{\boldsymbol{v}}) = 1 \quad (32)$$

In the example in the next section, we specify values for the real economy, popularity characteristics, investor wealth, and investor preferences for risk as well as popularity characteristics, and then numerically solve equations 28-32 for market values.[12]

A Numerical Example

To illustrate the PAPM with *heterogeneous* expectations, we created a simple numerical example with six securities, up to two popularity characteristics per security, and six investors that are designed in the spirt of a controlled experiment to highlight the incremental differences in the various models. While we could allow for heterogeneous expected values and variances and covariances for end-period security values, as well as heterogeneous views of the popularity characteristic of the securities, to keep the example manageable, the only heterogeneous view of a security that we vary across investors is the expected end-period value of two of the six securities. (As in IIKX, we also have heterogenous preferences for popularity characteristics.)

[12] Solving for the equilibrium values is arguably the most complicated aspect of the models. For the CAPM and PAPM respectively, Appendices B and C of Chapter 5 of IIKX, explain the process for solving for equilibrium values. For the CAPM there is a closed form solution, while the PAPM must be solved numerically. A companion spreadsheet is available from the authors that illustrates the calculations in the example.

Although the PAPM does not require an all-knowing Economist to achieve equilibrium, it is helpful to postulate one so that we can observe the "truth," i.e. the prices we would have if investors had rational expectations as defined by Muth (1961).[13] In our numerical example, by agreeing on the expected ending values and the "true prices" of securities, we can potentially measure the degree of inefficiencies or mispricing in the market and determine the portfolio misallocations.

Exhibit 3 presents our assumptions regarding the *true* expected ending values, standard deviations, correlations, and popularity characteristics/exposures of the six securities. In these numerical examples, we assume that there is an all-knowing Economist that knows the true expected ending values. We highlight the unique values. In Exhibit 3 notice that Security A is high risk and Security B is low risk. Next, Security E has a high level of Popularity Characteristic X and Security F has a high level of Popularity Characteristic Y.[14]

[13] Under Muth's definition of rational expectations, all agents in the model have the same correct expectations.
[14] In the real world, as documented in IIKX there are likely numerous popularity characteristics and most securities will have non-zero exposures or loadings to the identified characteristics.

Exhibit 3: **The Real Economy with Popularity Characteristics**

Security	Correct Expected Ending Value ($)	Standard Deviation of Ending Value ($)	Popularity Characteristic X	Popular Characteristic Y
A	10	2.30	0.0	0.0
B	10	0.87	0.0	0.0
C	10.5	1.62	0.0	0.0
D	9.5	1.62	0.0	0.0
E	10	1.62	0.6	0.0
F	10	1.62	0.0	0.6

*The correlation of the ending values between all pairs of securities is 0.5.
The risk-free rate is 3%.

Exhibit 4 presents our assumption regarding the investors. Notice that Investor 1 is relatively risk tolerant and Investor 2 is relatively risk averse. Additionally, Investor 3 is correct to be bullish on the 5% higher ending value of Security C and Investor 4 is correct to be bearish on the 5% lower ending value of Security D. All other investors have incorrect expectations regarding the ending values of these two securities. Finally, Investor 5 has a strong preference for Popularity Characteristic X and Investor 6 has a strong dislike for Popularity Characteristic Y. The preferences for popularity characteristics of 5% and -5% are in the same unit as expected returns. Investors 1 through 4 do not have preferences regarding popularity characteristics.

Exhibit 4: Investors

Investor	Fraction of Market Wealth (%)	Risk Aversion	Expected Ending Value Securities A, B, E, F ($)	Expected Ending Value of Security C ($)	Expected Ending Value of Security D ($)	Preference for Popularity Characteristic X (%)	Preference for Popularity Characteristic Y (%)
1	5.0	3	10.0	10.0	10.0	0.0	0.0
2	5.0	5	10.0	10.0	10.0	0.0	0.0
3	22.5	4	10.0	10.5	10.0	0.0	0.0
4	22.5	4	10.0	10.0	9.5	0.0	0.0
5	22.5	4	10.0	10.0	10.0	5.0	0.0
6	22.5	4	10.0	10.0	10.0	0.0	-5.0

We structured the assumptions so that when comparing the results of the different asset pricing models, we can isolate the effects of risk aversion, forecasting skill, and popularity preferences on expected returns and security prices. Specifically, looking across Exhibits 3 and 4 we see:

1) Security A is the only high-risk security and Investor 1 is the only investor with a low level of risk aversion. Similarly, Security B is the only low-risk security and Investor 2 is the only investor with a high level of risk aversion. The other four securities all have moderate risk and the other four investors all have moderate risk aversion.
2) Investor 3's expected ending value of Security C is correct and Investor 4's expected ending value of Security D is correct. Hence, Investor 3 has perfect skill regarding Security C and Investor 4 has perfect skill regarding Security D. All other investor estimates of expected ending values on Securities C and D are incorrect. (Note that all expected ending values on all of the other four securities are correct, and all of the investors have the correct level of uncertainty about actual ending values.)
3) Security E is the only security that exhibits Popularity Characteristic X and Investor 5 is the only investor that has a preference regarding Popularity Characteristic X. Similarly, Security F is the only security that exhibits Popularity Characteristic Y and Investor 6 is the only investor that has a preference regarding Popularity Characteristic Y, in this case, a negative preference. The specified preferences of 5% and -5% in this example are relatively extreme representing a very strong positive preference for Popularity Characteristic X and a very strong dislike for Popularity Characteristic Y.

We first solve the model under the assumptions of the CAPM with homogeneous forecasts, then again under the assumptions of the PAPM with heterogeneous forecasts.

Exhibits 5 and 6 show the results of solving the models.

Exhibit 5 shows:

1) The expected returns of all investors under the CAPM with homogeneous expectations. In this model, these are the expected returns of all investors.
2) The correct expected returns under the PAPM with heterogeneous expectations.
3) The expected returns of the four investors (1, 2, 5, and 6) who have the same correct expectations for the ending values of Securities A, B, E, and F, and incorrect expectations for the ending values of Securities C and D.
4) Investor 3's expected returns. These are correct except for Security D.
5) Investor 4's expected returns. These are correct except for Security C.
6) The market weighted average opinion of expected returns calculated using equation 5. These are correct for Securities A, B, E, and F. They are incorrect for Securities C and D due to the forecasting errors of Investors 1, 2, 4, 5, and 6 for Security C and the forecasting errors of Investors 1, 2, 3, 5, and 6 for Security D.

Exhibit 5: Expected Returns | CAPM with Homogeneous Expectations versus PAPM with Heterogeneous Expectations

Security	CAPM Homogeneous: Expected Returns (%)	PAPM Heterogeneous Expected Returns From Different Perspectives (%)				
		Correct	Investors 1, 2, 5, 6	Investor 3	Investor 4	Market Weighted Average
A	14.50	14.50	14.50	14.50	14.50	14.50
B	6.57	6.57	6.57	6.57	6.57	6.57
C	10.01	14.54	9.08	14.54	9.08	10.30
D	10.81	6.14	11.73	11.73	6.14	10.48
E	10.39	9.67	9.67	9.67	9.67	9.67
F	10.39	11.11	11.11	11.11	11.11	11.11

Moving from expected returns to portfolio holdings, Exhibit 6 shows:

1) The market portfolio under the CAPM with homogeneous expectations.
2) The portfolio of each of the six investors under the PAPM with heterogeneous expectations. Note:
    a. Investor 1 shorts the risk-free asset, to create a levered portfolio based on their low level of risk aversion.
    b. Investor 2 invests part of their wealth in the risk-free asset (Cash), to create a unlevered portfolio based on their high level of risk aversion.
    c. Investor 3's large allocation to Security C due to Investor 3's high level of expected return on Security C. (Security C is undervalued.)
    d. Investor 4's large short position in Security D due Investor 4's low level of the expected return on Security D. (Security D is overvalued.)
    e. Investor 5's large allocation to Security E due to Investor 5's preference for securities with exposure to Characteristic X, where Security E is the only security with such exposure.
    f. Investor 6's short position in Security F due to Investor 6's negative preference for securities with exposure to Characteristic Y, where Security F is the only security with such exposure.
3) The market portfolio under the PAPM with heterogeneous expectations, which is the aggregation of the portfolios of all market participants. In general, this is <u>not</u> on the efficient frontier drawn using the correct expectations.
4) The portfolio under the PAPM with heterogeneous expectations that maximizes the Sharpe ratio <u>is</u> on the efficient frontier drawn using correct expectations; however, no investor in this example has correct expectation across all six securities.

Exhibit 6: Portfolios | CAPM with Homogeneous Expectation versus PAPM with Heterogeneous Expectations

Security	CAPM Homogeneous: Portfolio Weights (%)	PAPM Heterogeneous: Portfolio Weights (%)							
		Investor 1	Investor 2	Investor 3	Investor 4	Investor 5	Investor 6	Market	Max. Sharpe Ratio
A	16.07	21.30	12.78	7.61	24.34	11.40	20.49	16.07	15.85
B	17.29	22.92	13.75	-6.55	40.93	4.20	30.01	17.26	17.05
C	17.56	-3.56	-2.13	71.89	9.76	-9.46	4.04	16.87	83.65
D	15.77	47.14	28.29	23.23	-37.43	28.72	41.91	16.47	-49.17
E	16.66	8.14	4.88	-6.25	18.46	46.67	12.78	16.78	6.06
F	16.66	35.70	21.42	14.58	38.97	20.10	-12.75	16.56	26.56
Cash	0.00	-31.65	21.01	-4.50	4.97	-1.63	3.52	0.00	0.00

Exhibit 7 shows the impact of heterogeneous expectations and popularity preferences on security prices. To isolate the impact of heterogenous expectations, we calculate the security prices under the CAPM with heterogenous expectations and compare them to the prices under the CAPM with homogeneous expectations. To isolate the impact of popularity preferences, we calculate security prices under the PAPM with heterogeneous expectations and compare them to the prices under the CAPM with heterogeneous expectations. Finally, we see the combined impact of heterogeneous expectations and popularity preferences, and we compare prices under the PAPM with heterogenous expectations to those under the CAPM under homogeneous expectations.

Exhibit 7: Impact of Heterogeneous Expectations and Popularity Preferences on Asset Prices

Security	CAPM (Hom) ($)	CAPM (Het) ($)	CAPM (Het) – CAPM (Hom) % Diff.	PAPM (Het) ($)	PAPM (Het) – CAPM (Het) % Diff.	PAPM (Het) – CAPM (Hom) % Diff
A	8.73	8.73	0.00	8.73	0.00	0.00
B	9.38	9.38	0.00	9.38	0.00	0.00
C	9.54	9.17	-3.95	9.17	0.00	-3.95
D	8.57	8.95	4.40	8.95	0.0	4.40
E	9.06	9.06	0.00	9.12	0.66	0.66
F	9.06	9.06	0.00	9.00	-0.65	-0.65

As Exhibit 7 shows, the incorrect expectations of five of the six investors in each case cause Securities C and D to be mispriced. Similarly, the popularity preferences of Investors 5 and 6 cause pricing biases for Securities E and F.[15] In this particular example, the impact of heterogeneous expectation on the expected returns of Securities C and D is much greater than the impact of the popularity preferences on Securities E and F. This is because five out of six of the investors have the wrong return expectations for Securities C and D respectively, thus their combined impact on price is significant. In contrast, only one of the six investors has a preference related to Security E and only one of the six investors has a preference related to Security F, thus the impact of the preferences is relatively low. These results are particular to this example.[16]

Exhibits 8—11 graphically show the expected returns and standard deviation of the securities and the investor portfolios under each of our four models; namely,

1) The CAPM with homogeneous expectations (Exhibit 8)
2) The CAPM with heterogeneous expectations (Exhibit 9)
3) The PAPM with homogeneous expectations (Exhibit 10)
4) The PAPM with heterogenous expectations (Exhibit 11)

All of the points shown are plotted using correct expectations.

In addition to the points representing the securities and investor portfolios, each exhibit includes:

1) The efficient frontier of risky securities
2) The market portfolio
3) The portfolio that maximizes the Sharpe ratio
4) The capital market line; i.e., the line that runs through the risk-free asset and that is tangent to the efficient frontier at the portfolio that maximizes the Sharpe ratio.

There are important and subtle differences across the four charts.

Starting with Exhibit 8, the CAPM with homogeneous expectations, the market portfolio and Sharpe maximizing portfolio are one and the same. Since Investors 3, 4, 5, and 6 all have the same level of risk-aversion that is the same as the market average level of risk aversion, they all hold the market portfolio / Sharpe ratio maximizing portfolio. Investor **1 / 2** have **lower** / **higher** than average risk-aversion and thus due to risk preference choose a **levered** / **unlevered** position in the market portfolio. Again, since all investors hold some portion of the market portfolio, optimization is not needed to form any of the portfolios.

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[15] The zeros shown in Exhibit 7 are only approximate, since changing preferences and expectations can have second-order effects on the resulting equilibrium values.

[16] This is not indicative of some greater truth around the impact or expectations versus popularity preferences. In an alternative example, we could reverse this result by allowing most of the investor to be correct and allowing most of the investors to have popularity preferences.

Looking across Exhibits 9, 10, and 11, the risk and expected return of at least two securities are different relative to what Exhibit 8 shows, and as such, the composition of each of the corresponding efficient frontiers are different. The three efficient frontiers result in Sharpe maximizing portfolios that no longer correspond with the original market portfolio nor the corresponding market portfolio in a given model. This leads to three distinct capital market lines all of which dominate the corresponding market portfolio and show that the market portfolio is not efficient when one incorporates either unique forecasts or popularity preferences. In the case of the PAPM with homogeneous expectations (Exhibit 10), we drew the marker for the market portfolio large enough to make it easily visible, although its true risk and expected return are at the center of the marker and thus beneath the tangency line.

Focusing on Exhibit 9, the CAPM with heterogeneous expectations, from Exhibits 3 and 4, we see that all six of the investors are making at least one forecasting error. Optimization is required by all six investors. Investors 1, 2, 5, and 6 have the same correct estimates of the expected ending values for securities A, B, E, and F, and these four investors have estimated the wrong expected ending values for both Securities C and D. Investor 3 is the only investor who recognizes Security C is underpriced and Investor 4 is the only investor who recognizes Security D is overpriced, as such, they are more skilled and their portfolios are closer to the efficient frontier.

Moving to Exhibit 10, the PAPM with homogeneous expectations, Investors 5 and 6 have distinct preferences for the popularity characteristics exhibited by Security E and F, respectively, which cause them to no longer hold the market portfolio forming unique portfolios reflecting their **popularity preferences**, resulting in an equilibrium that differs from that of the standard CAPM. Notice that the plot points of Security E and F (which have the same plot point in Exhibit 8), differ from what they are in Exhibit 8. Optimization is required by all investors, with Investor 5 and 6 arriving at unique portfolios and Investors 1, 2, 3, and 4 finding the Sharpe maximizing portfolio that Investors 1 and 2 lever / unlever. In this example as explained earlier, the impact of preferences is relatively small; thus, the market portfolio is close to the Sharpe maximizing portfolio, but slightly less efficient.

Finally in Exhibit 11, the PAPM with heterogeneous expectations, we can see the combined effects of **heterogeneous expectations** and **popularity preferences** simultaneously, that is the unique ending value estimates of Investors 3 and 4 for Security C and D as well as the popularity preferences of Investors 5 and 6 for Security E and F. Security C, D, E, and F all have plots points that differ from what they are in the standard CAPM (Exhibit 8) and no investor holds a portfolio on the capital market line. Optimization is required by all investors. Investors 3 and 4 only make one forecasting error and thus arrive at portfolios closer to the efficient frontier. Investors 5 and 6 have the same skill level as Investors 1 and 2 (they all have the same forecasts including two forecasting errors); yet, due to their popularity preferences form portfolios with the lowest Sharpe ratios.

Exhibit 8: The CAPM with Homogeneous Expectations

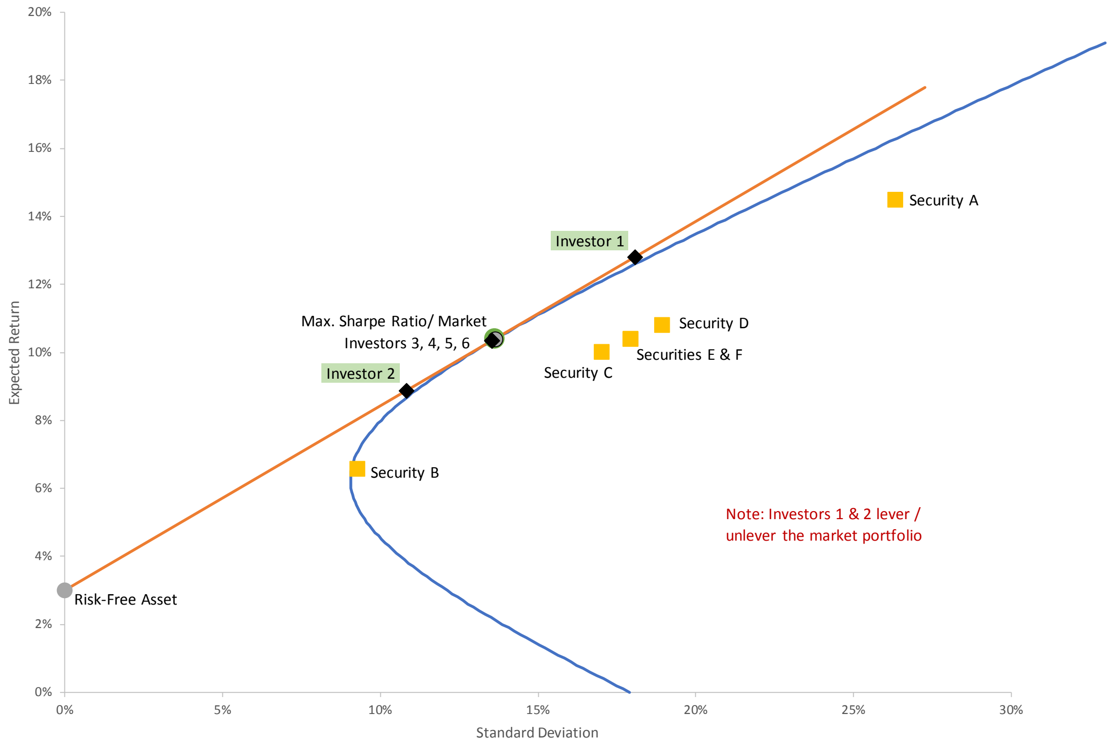


Exhibit 9: The CAPM with Heterogeneous Expectations

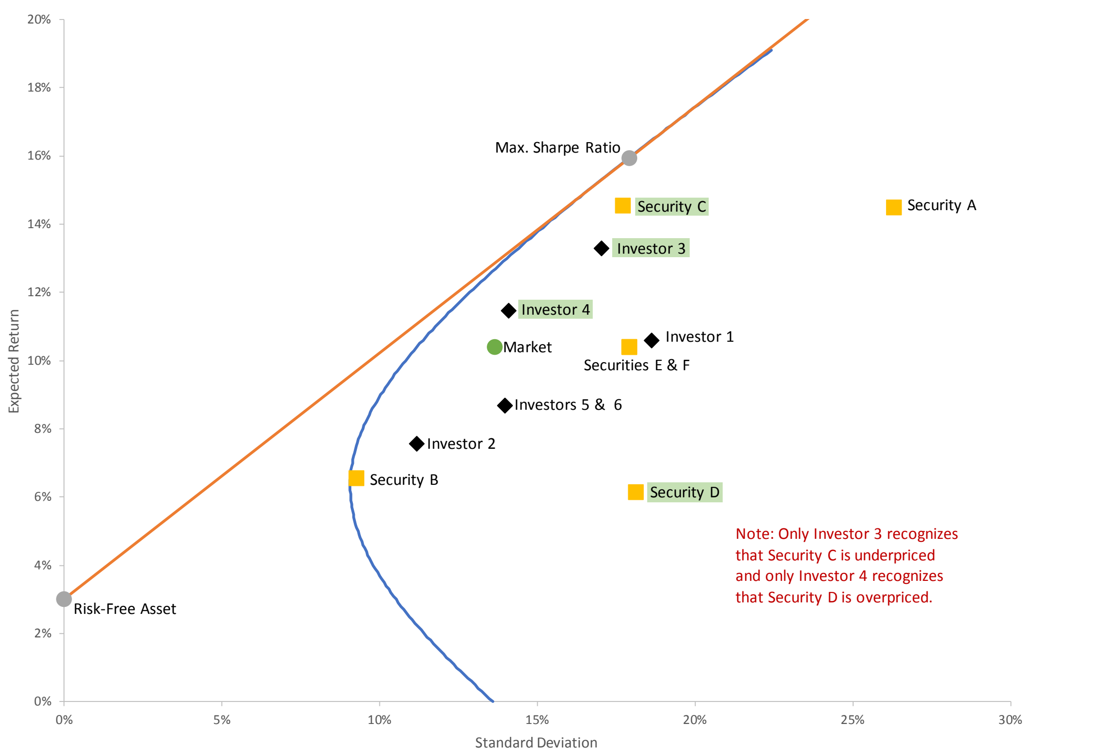


Exhibit 10: The PAPM with Homogeneous Expectations

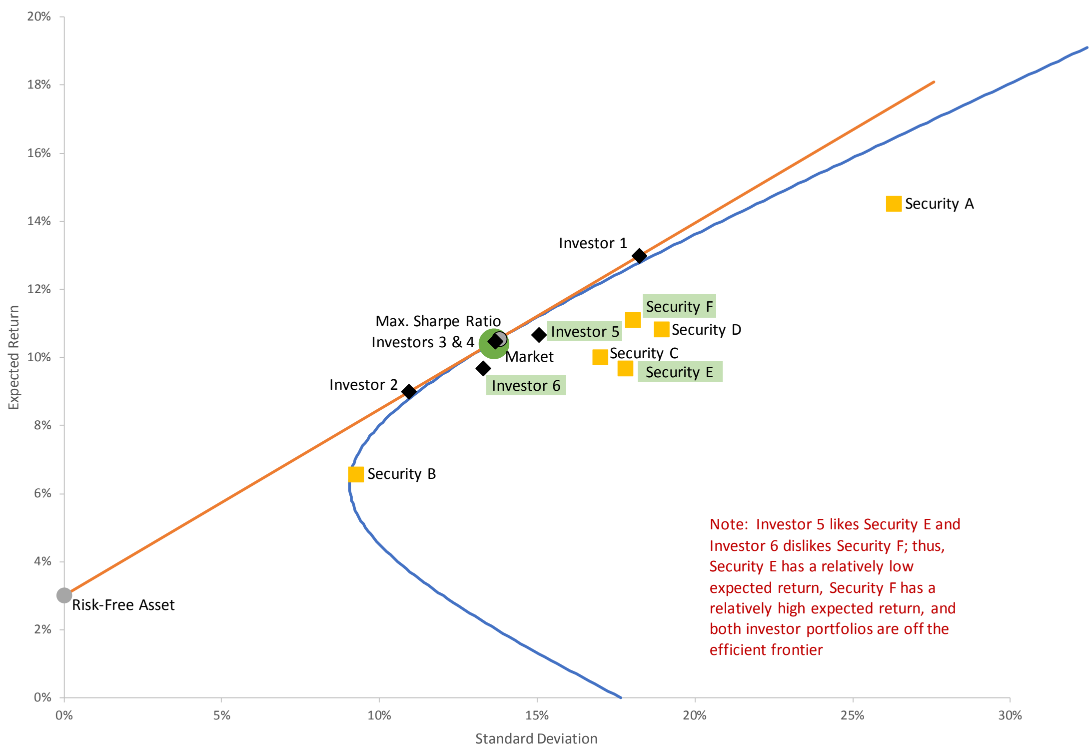
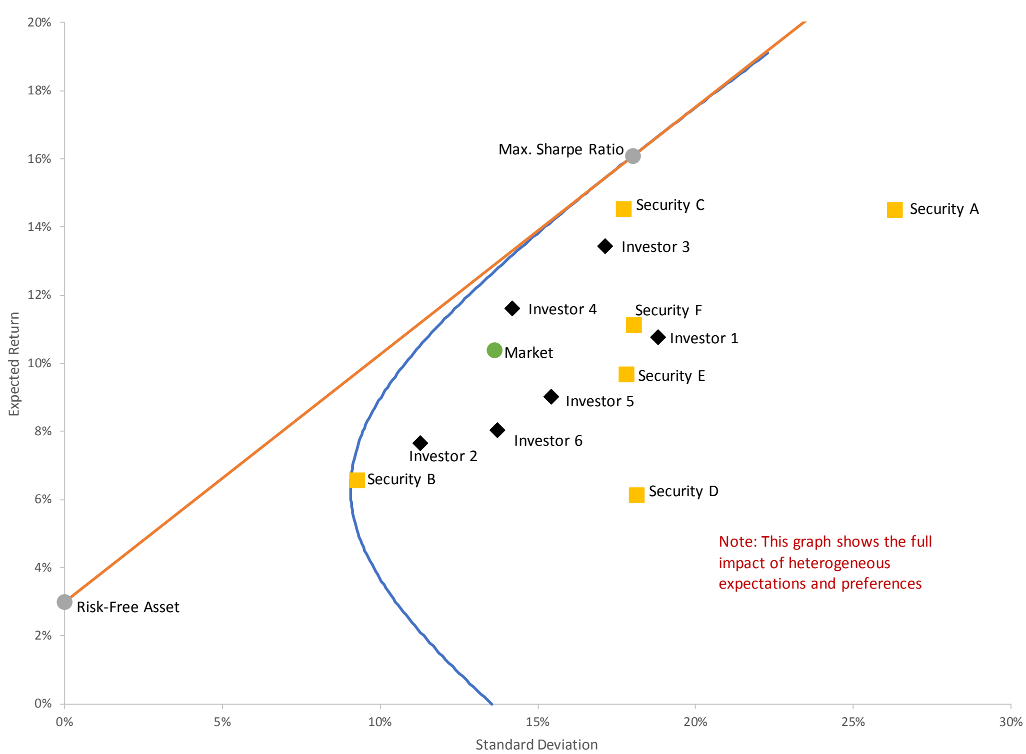


Exhibit 11: The PAPM with Heterogeneous Expectations



Conclusions

We present a formal version of the PAPM with heterogeneous expectations followed by a simplified, controlled example, that enables us to illustrate the differences between four models:

The CAPM with Homogeneous Expectations
The CAPM with Heterogeneous Expectations
The PAPM with Homogeneous Expectations
The PAPM with Heterogeneous Expectations

The original **CAPM / PAPM** with homogeneous expectations have very similar assumptions yet due to the PAPM's ability to consider investor preferences beyond risk, the two models lead to very different conclusions.

Relaxing the CAPM assumption that all investors share the same forecasts, leads to investors needing optimization to arrive at a unique portfolio that reflects their unique forecasts. This is vastly different from the standard CAPM world in which all investors hold a portion of their wealth in the market portfolio. Although each investor has his or her own security expectations leading them to unique portfolios, the CAPM still holds at the aggregate level for each security and for the market as a whole. Relaxing the same assumption for the PAPM, has no major impact on the conclusions of the model, only that the rationale for holding a unique portfolio is expanded beyond unique preferences to include unique forecasts. Most of the elegance and ease of the CAPM falls apart at the individual investor level when you allow heterogeneous forecasts, while the PAPM is nearly unaffected.

Heterogeneous expectations lead to mispricing. Since investors are not privy to the knowledge of the all-knowing Economist outside of the model, investors cannot observe mispricing relative to the true price, even though mispricing is occurring.

For both the CAPM and the PAPM, allowing heterogeneous forecasts means that not all investors have rational expectations, making the market less than perfectly efficient and giving rise to various types of market participants, some of which can be considered losers or winners.

Finally, the PAPM with heterogeneous expectations can be viewed as an extension of the CAPM, but it may be more useful to view the CAPM as special case of the PAPM. The PAPM is general in that it allows for rational investors, irrational investors, and everyone in between; it allows for efficient markets, inefficient markets, and a variety of efficiency levels; and finally, it allows for investors with and without unique risk and non-risk preferences. By incorporating both investor preferences and allowing for heterogeneous expectations, the PAPM takes two major steps towards realistic asset pricing assumptions and provides a bridge between classical and behavioral finance.

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